5. **Exact Equations** If a DE of the form : $M(x;y)dx + N(x;y)dy = 0$ has $M_y = N_x$ where G_x represents the partial derivative of a function G with respect x then, the equation is called exact. If this is the form then we can find $F(x;y)$ such that $F_x = M(x;y)$ and $F_y = N(x;y)$.

Example 3.

$$(3y\cos(x) + 4xe^x + 2x^2e^x)dx + (3\sin(x) + 3)dy = 0$$
$$M = 3y\cos(x) + 4xe^x + 2x^2e^x \implies M_y = 3\cos(x)$$
$$N = 3\sin(x) + 3 \implies N_y = 3\cos(x)$$

Hence the equation is exact. $F_x = 3y\cos(x) + 4xe^x + 2x^2e^x$ *is difficult to integrate, so choose,* $F_y = 3\sin(x) + 3$ *which is easy.*

$$\int F_y dy = \int (3\sin(x) + 3)dy$$

$$F = 3y\sin(x) + 3y + h(x) + C$$

Now need to find $h(x)$*.*

$$F_x = 3y\cos(x) + h'(x) = 3y\cos(x) + 4xe^x + 2x^2e^x$$
$$h'(x) = 4xe^x + 2x^2e^x$$

This is a special integral of the form $\int e^x(f(x) + f'(x)) = e^x f(x)$

$$h(x) = 2x^2e^x$$

$\implies F(x;y) = 3y\sin(x) + 3y + 2x^2e^x + C$ *which is the solution to the DE.*

6. **Almost Exact Equations** If $M_y \neq N_x$, there is still hope. We can use integrating factor μ such that, if $q(x) = \frac{M_y - N_x}{N}$ is independent of y or if $p(y) = \frac{N_x - M_y}{M}$ is independent of x then $\mu(x) = e^{\int q(x)}$ if the first condition is met or $\mu(y) = e^{\int p(y)}$ if the second condition is met. If both are met $\mu(x;y) = q(x) \cdot p(y)$.

Example 4.

$$(27xy^2 + 8y^3)dx + (18x^2y + 12xy^2)dy = 0$$

The fact that equation is not exact could be easily verified.

$$M_y = 54xy + 24y^2 \quad \& \quad N_x = 36xy + 12y^2$$

situation for $\frac{N_x - M_y}{M}$ *is left as an exercise.*

$$q(x) = \frac{M_y - N_x}{N}$$
$$q(x) = \frac{18xy + 12y^2}{18x^2y + 12xy^2} = \frac{y(18x + 12y)}{xy(18x + 12y)} = \frac{1}{x}$$
$$\mu(x) = e^{\int \frac{1}{x}} = x$$

Multiply the DE by x.

$$(27x^2y^2 + 8xy^3)dx + (18x^3y + 12x^2y^2)dy = 0$$

This DE is exact and can be solved as shown in the above section.